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Loop equations, matrix models, and $\mathcal{N}=1$ supersymmetric gauge theories

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Abstract: We derive the Konishi anomaly equations for $\mathcal{N}=1$ supersymmetric gauge theories based on the classical gauge groups with matter in two-index tensor and fundamental representations, thus extending the existing results for $\mathrm{U}(N)$. A general formula is obtained which expresses solutions to the Konishi anomaly equation in terms of solutions to the loop equations of the corresponding matrix model. This provides an alternative to the diagrammatic proof that the perturbative part of the glueball superpotential W_{eff} for these matter representations can be computed from matrix model integrals, and further shows that the two approaches always give the same result. The anomaly approach is found to be computationally more efficient in the cases we studied. Also, we show in the anomaly approach how theories with a traceless two-index tensor can be solved using an associated theory with a traceful tensor and appropriately chosen coupling constants.

Keywords: Nonperturbative Effects, Gauge Symmetry, Supersymmetric Effective Theories, Matrix Models.

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Contents

1. Introduction

The recently established connection [1, 2, 3] between matrix models and the effective superpotentials of certain $\mathcal{N} = 1$ gauge theories provides us with a new tool for studying supersymmetric field theories. The connection, originally formulated in the context of U(N) gauge theories with adjoint matter, has been established following two distinct approaches, one based on superspace diagrammatics [4], and the other on generalized Konishi anomalies [5]. These derivations were subsequently generalized to a few more gauge groups and matter representations, but the list of examples is actually quite short at present. In particular, the diagrammatic approach has been applied to the classical gauge groups with matter in arbitrary two-index representations [7]–[11], while the anomaly approach has so far been used for U(N) with matter in the adjoint and fundamental representations [5, 16], in (anti)symmetric tensor representations [11], and to quiver theories [11, 12].[1] So basic questions remain regarding the general applicability of these ideas, and also whether matrix models can in fact successfully reproduce the known physics of supersymmetric gauge theories.

[1] The Konishi anomalies have also been applied without direct reference to a matrix model in [14].

In [10], theories based on the classical gauge groups with two-index tensor matter were considered using the diagrammatic approach. In the case[2] of $\mathrm{Sp}(N)$ with anti-symmetric matter, a comparison was made against an independently derived dynamical superpotential [13] governing these theories. The comparison revealed agreement up to $h-1$ loops in perturbation theory (h is the dual Coxeter number), and a disagreement at h loops and beyond. Although it seemed most likely that the disagreement was due to nonperturbative effects, even at the perturbative level there were a number of subtleties deserving of further scrutiny. These subtleties mainly concern the class of diagrams which should be kept in the evaluation of the superpotential, and whether one is allowed to use Lie algebra identities to express objects of the form $\mathrm{Tr}(\mathcal{W}_\alpha)^{2h}$ in terms of lower traces including the glueball superfield $S \sim \mathrm{Tr}(\mathcal{W}_\alpha)^2$. Since these subtleties arise at the same order in perturbation theory as the observed discrepancies, it seems important to gain a better understanding of them. One motivation for the present work was to rederive the results of [10] in the anomaly approach to see if this gives the same result, and if so, to see which diagrams are effectively being computed. We will see that the anomaly approach corresponds to keeping at most two $\mathcal{W}_\alpha$'s per index loop and not using Lie algebra identities. So using these rules, whether one computes using diagrams or anomalies, one finds the same agreements/discrepancies between the gauge theory and the matrix model.

Another motivation for this work was to apply the anomaly approach to a wider class of theories. For the classical gauge groups with certain two-index tensors plus fundamentals, we will show how solutions to the Konishi anomaly equations can be obtained from solutions to the loop equations of the corresponding matrix model. This leads to the following general formula for the perturbative contribution to the effective glueball superpotential

$$W_{\mathrm{eff}} = N\frac{\partial}{\partial S}\mathcal{F}_{S^2} + \frac{w_\alpha w^\alpha}{2}\frac{\partial^2}{\partial S^2}\mathcal{F}_{S^2} + 4\mathcal{F}_{RP^2} + \mathcal{F}_{D^2} \tag{1.1}$$

where the $\mathcal{F}$'s are matrix model contributions of a given topology to the free energy. This formula generalizes the $\mathrm{U}(N)$ results of [5, 16, 11], as well as results [7]–[11] found using the diagrammatic approach.

In fact, the above formula is only directly applicable to cases in which no tracelessness condition is imposed on the two-index tensors. In [10] it was shown that imposing a tracelessness condition requires one to include additional disconnected matrix model diagrams, and there was no simple formula relating the superpotential to the free energy of the traceless matrix model. On the other hand, one expects that the traceful theory should contain all the information about the traceless case provided one includes a Lagrange multiplier field to set the trace to zero. We will show how this works in detail, and find that indeed, the superpotential of the traceless theory can be extracted from the free energy of the traceful matrix model. We use this to rederive and extend some results from [10] in a much more convenient fashion.

The remainder of this paper is organized as follows. In sections 2 and 3 we derive the gauge theory Konishi anomaly equations and the matrix model loop equations for the theories of interest. The theories can all be treated in a uniform way by using appropriate

[2] Our convention is such that $\mathrm{Sp}(2) \approx \mathrm{SU}(2)$.

projection operators. In section 4 we discuss some of the subtleties alluded to above, and then go on to show that solutions to the gauge theory anomaly equations follow from those of the matrix model loop equations. Section 5 concerns the effects of tracelessness. Details of some of our calculations are given in appendices A and B.

Note: As we were preparing the manuscript, [24] appeared which overlaps with some of our discussion.

2. Loop equations on the gauge theory side

In this section we derive the gauge theory loop equations for various gauge groups and matter representations, extending the U(N) result of [5, 16].

2.1 Setup

We consider an $\mathcal{N}=1$ supersymmetric gauge theory with tree level superpotential

$$W_{\text{tree}} = \mathrm{Tr}[W(\Phi)] + \tilde{Q}_{\tilde{f}} m_{\tilde{f}f}(\Phi) Q_f \,, \tag{2.1}$$

where the two-index tensor Φ_{ij} is in one of the following representations:

- U(N) adjoint.
- SU(N) adjoint.
- SO(N) antisymmetric tensor.
- SO(N) symmetric tensor, traceful or traceless.
- Sp(N) symmetric tensor.
- Sp(N) antisymmetric tensor, traceful or traceless.

For other U(N) representations, see [17, 11]. In the Sp cases, the object with the denoted symmetry is related to Φ by

$$\Phi = \begin{cases} SJ & S_{ij}\text{: symmetric tensor}\,, \\ AJ & A_{ij}\text{: antisymmetric tensor}\,. \end{cases} \tag{2.2}$$

Here J is the invariant antisymmetric tensor of Sp(N), namely

$$J_{ij} = \begin{pmatrix} 0 & \mathbb{1}_{N/2} \\ -\mathbb{1}_{N/2} & 0 \end{pmatrix}. \tag{2.3}$$

The tracelessness of the Sp antisymmetric tensor is defined with respect to this J, i.e., by $\mathrm{Tr}[AJ]=0$.

Also, Q_f and $\tilde{Q}_f$ are fundamental matter fields, with f and $\tilde{f}$ being flavor indices. In the U(N) case we have N_f fundamentals Q_f and N_f anti-fundamentals $\tilde{Q}_{\tilde{f}}$, while in the SO / Sp case we have N_f fundamentals Q_f. In the SO / Sp case, $\tilde{Q}_{\tilde{f}}$ is not an independent field but related to Q_f by

$$(\tilde{Q}_{\tilde{f}})_i = \begin{cases} (Q_{\tilde{f}})_i & \mathrm{SO}(N)\,, \\ (Q_{\tilde{f}})_j J_{ji} & \mathrm{Sp}(N)\,. \end{cases} \tag{2.4}$$

In the Sp case, N_f should be taken to be even to avoid the Witten anomaly [6].

W and m are taken to be polynomials

$$W(z) = \sum_{p=1}^{n} \frac{g_p}{p} z^p\,, \qquad m_{\tilde{f}f}(z) = \sum_{p=1}^{n'} \frac{(m_p)_{\tilde{f}f}}{p} z^p\,, \tag{2.5}$$

where in the traceless cases the $p = 1$ term is absent from $W(z)$. Further, due to the symmetry properties of the matrix Φ, some g_p vanish for certain representations:

$$g_{2p+1} = 0 \quad (p = 0, 1, 2, \cdots) \qquad \text{for SO antisymmetric / Sp symmetric.} \tag{2.6}$$

The symmetry properties of Φ also imply that the matrices $(m_p)_{\tilde{f}f}$ have the following symmetry properties:

$$(m_p)_{f'f} = \begin{cases} (-1)^p (m_p)_{ff'} & \text{SO antisymmetric}\,, \\ (m_p)_{ff'} & \text{SO symmetric}\,, \\ (-1)^{p+1}(m_p)_{ff'} & \text{Sp symmetric}\,, \\ -(m_p)_{ff'} & \text{Sp antisymmetric}\,. \end{cases} \tag{2.7}$$

In this and the next few sections, we discuss traceful cases only, postponing the traceless cases to section 5 (we regard the SU(N) case as the traceless U(N) case).

2.2 The loop equations

We will be interested in expectation values of chiral operators. As in [5, 16],

$$\{\mathcal{W}_\alpha, \mathcal{W}_\beta\} = [\Phi, \mathcal{W}_\alpha] = \mathcal{W}_\alpha Q = \tilde{Q}\mathcal{W}_\alpha = 0 \tag{2.8}$$

in the chiral ring. Therefore, the complete list of independent single-trace chiral operators are $\mathrm{Tr}[\Phi^p]$, $\mathrm{Tr}[\mathcal{W}_\alpha \Phi^p]$, $\mathrm{Tr}[\mathcal{W}^2\Phi^p]$, and $\tilde{Q}_{\tilde{f}}\Phi^p Q_f$. As is standard, we define

$$S = -\frac{1}{32\pi}\,\mathrm{Tr}[\mathcal{W}_\alpha \mathcal{W}^\alpha]\,, \qquad w_\alpha = \frac{1}{4\pi}\,\mathrm{Tr}[\mathcal{W}_\alpha]\,. \tag{2.9}$$

The chiral operators can be packaged concisely in terms of the resolvents

$$\begin{aligned} R(z) &\equiv -\frac{1}{32\pi^2}\left\langle \mathrm{Tr}\left[\frac{\mathcal{W}^2}{z-\Phi}\right]\right\rangle, \qquad & w_\alpha(z) &\equiv \frac{1}{4\pi}\left\langle \mathrm{Tr}\left[\frac{\mathcal{W}_\alpha}{z-\Phi}\right]\right\rangle, \\ T(z) &\equiv \left\langle \mathrm{Tr}\left[\frac{1}{z-\Phi}\right]\right\rangle, \qquad & M_{f\tilde{f}}(z) &\equiv \left\langle \tilde{Q}_{\tilde{f}}\frac{1}{z-\Phi}Q_f\right\rangle. \end{aligned} \tag{2.10}$$

Note that the indices of $M_{f\tilde{f}}$ are reversed relative to $\tilde{Q}_{\tilde{f}}$, Q_f. The resolvent $w_\alpha(z)$ is nonvanishing only for U(N); in all other cases $w_\alpha(z) \equiv 0$. This can be understood as follows. In these semi-simple cases the Lie algebra generators are traceless, so we cannot have a nonzero background field w_α. There being no preferred spinor direction specified by the background w_α, the spinor $w_\alpha(z)$ can be nothing but zero. Alternatively, if we integrate out Φ, then $w_\alpha(z)$ should be of the form $\left\langle \mathrm{Tr}[\mathcal{W}_\alpha](\mathrm{Tr}[\mathcal{W}^2])^n \right\rangle$ by the chiral ring relations (2.8). If we use the factorization property of chiral operator expectation values, this is proportional to $w_\alpha S^n$, which vanishes.

The resolvents defined in equation (2.10) provide sufficient data to determine the effective superpotential up to a coupling independent part, because of the relation

$$\langle \mathrm{Tr}[\Phi^p] \rangle = p \frac{\partial}{\partial g_p} W_{\mathrm{eff}} \,, \qquad \left\langle \tilde{Q}_{\tilde{f}} \Phi^p Q_f \right\rangle = p \frac{\partial}{\partial (m_p)_{\tilde{f}f}} W_{\mathrm{eff}} \,. \tag{2.11}$$

The generalized Konishi [21] anomaly equation [5, 16, 14] is obtained by considering the divergence of the current associated with the variation of a particular field Ψ_a:

$$\delta \Psi_a = f_a \,, \tag{2.12}$$

where a is a gauge index. Then the anomaly equation reads

$$\left\langle \frac{\partial W_{\mathrm{tree}}}{\partial \Psi_a} f_a \right\rangle + \frac{1}{32\pi^2} \left\langle [\mathcal{W}_\alpha \mathcal{W}^\alpha]_a^b \frac{\partial f_b}{\partial \Psi_a} \right\rangle = 0 \,, \tag{2.13}$$

where $\mathcal{W}_\alpha$ is in the representation furnished by Ψ. The first term in (2.13) represents the classical change of the action under the variation (2.12), while the second term in (2.13) corresponds to the quantum variation due to the change in the functional measure.

In the U(N) case considered in [5, 16], there is no additional symmetry imposed on the field Φ, so $\delta\Phi_{ij} = f_{ij}$ can be any function of $\mathcal{W}_\alpha$ and Φ. In general, the tensor Φ will have some symmetry properties (symmetric or antisymmetric tensor in the present SO / Sp study), and f_{ij} should be chosen to reflect those. Similarly, the derivative $\partial/\partial\Psi_a = \partial/\partial\Phi_{ij}$ should be defined in accord with the symmetry property of Φ_{ij}. To this end, we define a projector P appropriate to each case:

$$P_{ij,kl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \sigma t_{il} t_{jk}) \,, \tag{2.14}$$

where

$$\begin{cases} t_{ij} = \delta_{ij} \,, & \sigma = -1 \quad \text{SO antisymmetric} \,, \\ t_{ij} = \delta_{ij} \,, & \sigma = +1 \quad \text{SO symmetric} \,, \\ t_{ij} = J_{ij} \,, & \sigma = -1 \quad \text{Sp symmetric} \,, \\ t_{ij} = J_{ij} \,, & \sigma = +1 \quad \text{Sp antisymmetric} \;. \end{cases} \tag{2.15}$$

The tensor Φ_{ij} satisfies $P_{ij,kl}\Phi_{kl} = \Phi_{ij}$. Then, the symmetry property of $\delta\Phi$ discussed above is implemented by the replacements

$$f_a = f_{ij} \to P_{ij,kl} f_{kl} \,, \qquad \frac{\partial}{\partial \Psi_a} = \frac{\partial}{\partial \Phi_{ij}} \to P_{ij,kl} \frac{\partial}{\partial \Phi_{kl}} \,. \tag{2.16}$$

With this replacement, f_{ij} can be any function of $\mathcal{W}_\alpha$ and Φ as in the U(N) case. The derivative can be treated as in the U(N) case also.

There is no such issue for the Q and $\tilde{Q}$ fields, although we have to remember that they are not independent for SO / Sp.

With the projectors in hand, there is no difficulty in deriving the loop equations for SO / Sp. Here we just present the resulting loop equations, leaving the details to appendix A:

$$\begin{aligned}
[W'R]_- &= \frac{1}{2}R^2\,,\\
[W'T+\mathrm{tr}(m'M)]_- &= \begin{cases}\left(T-\frac{2}{z}\right)R & \text{SO antisymmetric}\,,\\ \left(T-2\frac{d}{dz}\right)R & \text{SO symmetric}\,,\\ \left(T+\frac{2}{z}\right)R & \text{Sp symmetric}\,,\\ \left(T+2\frac{d}{dz}\right)R & \text{Sp antisymmetric}\,,\end{cases}\\
2\left[(Mm)_{ff'}\right]_- &= R\delta_{ff'}\,,\\
2\left[(mM)_{\tilde{f}\tilde{f}'}\right]_- &= R\delta_{\tilde{f}\tilde{f}'}\,,
\end{aligned} \tag{2.17}$$

where $[F(z)]_-$ means to drop non-negative powers in a Laurent expansion in z. The last two equations are really the same equation due to the symmetry properties of m (see equation (2.7)), and Φ. Note that there is no $w_\alpha(z)$ in these cases as explained below eq. (2.10). For the sake of comparison, the U(N) loop equations are [5, 16]

$$\begin{aligned}
[W'R]_- &= R^2\,,\\
[W'w_\alpha]_- &= 2w_\alpha R\,,\\
[W'T+\mathrm{tr}(m'M)]_- &= 2TR+w_\alpha w^\alpha\,,\\
[(Mm)_{ff'}]_- &= R\delta_{ff'}\,,\\
[(mM)_{\tilde{f}\tilde{f}'}]_- &= R\delta_{\tilde{f}\tilde{f}'}\,.
\end{aligned} \tag{2.18}$$

One observes some extra numerical factors in the SO / Sp case as compared to the U(N) case. The $\frac{1}{2}$ in the first equation is from the $\frac{1}{2}$ in the definition of $P_{ij,kl}$, while the factor 2 in the last two equations is because in the SO / Sp case Q and $\tilde{Q}$ are really the same field, so the variation of $\tilde{Q}mQ$ under δQ for SO /Sp is twice as large as that for U(N). Finally, the $\frac{1}{z}R(z)$ and $\frac{d}{dz}R(z)$ terms in the second equation of (2.17) come from the second term of $P_{ij,kl}$.

The solution to the loop equations (2.17) or (2.18) is determined uniquely [5] given the condition

$$S=\oint_C\frac{dz}{2\pi i}R(z)\,,\qquad w_\alpha=\oint_C\frac{dz}{2\pi i}w_\alpha(z)\,,\qquad N=\oint_C\frac{dz}{2\pi i}T(z)\,, \tag{2.19}$$

where the second equation is only for the U(N) case. The contour C goes around the critical point of $W(z)$. Therefore, if we recall the relation (2.11), we can say that the loop equations are all we need to determine the superpotential W_{eff}.

3. Loop equations on the matrix model side

Let us consider the matrix model which corresponds to the gauge theory in the previous section. Its partition function is

$$\boldsymbol{Z} = e^{-\frac{1}{\boldsymbol{g}^2}\boldsymbol{F}(\boldsymbol{S})} = \int d\boldsymbol{\Phi} d\boldsymbol{Q} d\tilde{\boldsymbol{Q}}\, e^{-\frac{1}{\boldsymbol{g}}W_{\rm tree}(\boldsymbol{\Phi},\boldsymbol{Q},\tilde{\boldsymbol{Q}})}\,. \tag{3.1}$$

We denote matrix model quantities by boldface letters. Here, $\boldsymbol{\Phi}$ is an $\boldsymbol{N}\times\boldsymbol{N}$ matrix with the same symmetry property as the corresponding matter field in the gauge theory. $\boldsymbol{Q}_f$ and $\tilde{\boldsymbol{Q}}_{\tilde{f}}$ are defined in a similar way to their gauge theory counterparts (therefore $d\tilde{\boldsymbol{Q}}$ in (3.1) is not included for SO / Sp). The function (or the "action") $W_{\rm tree}$ is the one defined in (2.1). We will take the $\boldsymbol{N}\to\infty$, $\boldsymbol{g}\to 0$ limit with the 't Hooft coupling $\boldsymbol{S}=\boldsymbol{g}\boldsymbol{N}$ kept fixed. The dependence of the free energy $\boldsymbol{F}(\boldsymbol{S})$ on $\boldsymbol{N}$ is eliminated using the relation $\boldsymbol{N}=\boldsymbol{S}/\boldsymbol{g}$, and we expand $\boldsymbol{F}(\boldsymbol{S})$ as

$$\boldsymbol{F}(\boldsymbol{S}) = \sum_{\mathcal{M}} \boldsymbol{g}^{2-\chi(\mathcal{M})}\mathcal{F}_{\mathcal{M}}(\boldsymbol{S}) = \mathcal{F}_{S^2} + \boldsymbol{g}\mathcal{F}_{RP^2} + \boldsymbol{g}\mathcal{F}_{D^2} + \cdots\,, \tag{3.2}$$

where the sum is over all compact topologies $\mathcal{M}$ of the matrix model diagrams written in the 't Hooft double-line notation, and $\chi(\mathcal{M})$ is the Euler number of $\mathcal{M}$. The cases which will be of interest to us are the sphere S^2, projective plane RP^2, and disk D^2, with $\chi=2,1,$ and 1, respectively. All other contributions have $\chi\le 0$.

We define matrix model resolvents as follows:

$$\boldsymbol{R}(z) \equiv \boldsymbol{g}\left\langle {\rm Tr}\left[\frac{1}{z-\boldsymbol{\Phi}}\right]\right\rangle, \qquad \boldsymbol{M}_{f\tilde{f}}(z) \equiv \boldsymbol{g}\left\langle \tilde{\boldsymbol{Q}}_{\tilde{f}}\frac{1}{z-\boldsymbol{\Phi}}\boldsymbol{Q}_f\right\rangle. \tag{3.3}$$

These resolvents provide sufficient data to determine the free energy $\boldsymbol{F}$ up to a coupling independent part since

$$\boldsymbol{g}\left\langle {\rm Tr}[\boldsymbol{\Phi}^p]\right\rangle = p\frac{\partial}{\partial g_p}\boldsymbol{F}\,, \qquad \boldsymbol{g}\left\langle \tilde{\boldsymbol{Q}}_{\tilde{f}}\boldsymbol{\Phi}^p\boldsymbol{Q}_f\right\rangle = p\frac{\partial}{\partial (m_p)_{\tilde{f}f}}\boldsymbol{F}\,. \tag{3.4}$$

We expand the resolvents in topologies just as we did for $\boldsymbol{F}$:

$$\boldsymbol{R}(z) = \sum_{\mathcal{M}} \boldsymbol{g}^{2-\chi(\mathcal{M})}\boldsymbol{R}_{\mathcal{M}}(z)\,, \qquad \boldsymbol{M}(z) = \sum_{\mathcal{M}} \boldsymbol{g}^{2-\chi(\mathcal{M})}\boldsymbol{M}_{\mathcal{M}}(z)\,. \tag{3.5}$$

Although $\boldsymbol{R}_{RP^2}$ and $\boldsymbol{R}_{D^2}$ are of the same order in $\boldsymbol{g}$, they can be distinguished unambiguously because all terms in $\boldsymbol{R}_{D^2}$ contains coupling constants $m_{\tilde{f}f}$, while $\boldsymbol{R}_{RP^2}$ does not depend on them at all. This is easily seen in the diagrammatic expansion of $\boldsymbol{F}$. Also, because $\boldsymbol{F}_{S^2}$ and $\boldsymbol{F}_{RP^2}$ do not contain m, the expansion of $\boldsymbol{M}$ starts from the disk contribution, $\boldsymbol{M}_{D^2}$.

Now we can derive the matrix model loop equations. Consider changing the integration variables as

$$\delta\boldsymbol{\Psi}_a = \boldsymbol{f}_a\,. \tag{3.6}$$

Since the partition function is invariant under this variation, we obtain

$$0 = -\frac{1}{\boldsymbol{g}}\frac{\partial W_{\rm tree}}{\partial\boldsymbol{\Psi}_a}\boldsymbol{f}_a + \frac{\partial\boldsymbol{f}_a}{\partial\boldsymbol{\Psi}_a}\,. \tag{3.7}$$

The first term came from the change in the "action" and corresponds to the first term (the classical variation) of the generalized Konishi anomaly equation (2.13). On the other hand, the second term came from the jacobian and corresponds to the second term (the anomalous variation) of eq. (2.13).

The derivation of the loop equations now can be done exactly in parallel to the derivation of the gauge theory loop equations. In the SO / Sp case, we again have to consider the projector $P_{ij,kl}$. Here we leave details of the derivation to appendix B and present the results. For SO / Sp, they are

$$\begin{aligned}
\boldsymbol{g}\left\langle \mathrm{Tr}\,\frac{W'(\boldsymbol{\Phi})}{z-\boldsymbol{\Phi}}\right\rangle + \boldsymbol{g}\left\langle \tilde{\boldsymbol{Q}}\frac{m'(\boldsymbol{\Phi})}{z-\boldsymbol{\Phi}}\boldsymbol{Q}\right\rangle &= \frac{1}{2}\left\langle\left(\boldsymbol{g}\,\mathrm{Tr}\,\frac{1}{z-\boldsymbol{\Phi}}\right)^2\right\rangle \pm \frac{\sigma}{2}\boldsymbol{g}^2\left\langle \mathrm{Tr}\,\frac{1}{(z-\boldsymbol{\Phi})(z-\sigma\boldsymbol{\Phi})}\right\rangle, \\
2\left\langle \tilde{\boldsymbol{Q}}_{\tilde{f}}\frac{m_{\tilde{f}f}(\boldsymbol{\Phi})}{z-\boldsymbol{\Phi}}\boldsymbol{Q}_{f'}\right\rangle &= \boldsymbol{g}\left\langle \mathrm{Tr}\,\frac{1}{z-\boldsymbol{\Phi}}\right\rangle \delta_{ff'}\,, \\
2\left\langle \tilde{\boldsymbol{Q}}_{\tilde{f}}\frac{m_{\tilde{f}'f}(\boldsymbol{\Phi})}{z-\boldsymbol{\Phi}}\boldsymbol{Q}_{f}\right\rangle &= \boldsymbol{g}\left\langle \mathrm{Tr}\,\frac{1}{z-\boldsymbol{\Phi}}\right\rangle \delta_{\tilde{f}\tilde{f}'}\,,
\end{aligned} \tag{3.8}$$

in the SO and Sp cases, respectively. The last two equations are really the same because of the symmetry properties of $\boldsymbol{\Phi}$ and $m_{\tilde{f}f}$.

Equations (3.8) include terms of all orders in $\boldsymbol{g}$. Expanding the matrix model expectation values in powers of $\boldsymbol{g}$, plugging in the expansion (3.5) and comparing the $\mathcal{O}(1)$ and $\mathcal{O}(\boldsymbol{g}^1)$ terms, we obtain the SO / Sp loop equations.[3] This is done in appendix B, and the results are:

$$\begin{aligned}
[W'\boldsymbol{R}_{S^2}]_- &= \frac{1}{2}(\boldsymbol{R}_{S^2})^2 \\
[W'\boldsymbol{R}_{RP^2}]_- &= \begin{cases} \left(\boldsymbol{R}_{RP^2} - \frac{1}{2z}\right)\boldsymbol{R}_{S^2} & \text{SO antisymmetric} \\ \left(\boldsymbol{R}_{RP^2} - \frac{1}{2}\frac{d}{dz}\right)\boldsymbol{R}_{S^2} & \text{SO symmetric} \\ \left(\boldsymbol{R}_{RP^2} + \frac{1}{2z}\right)\boldsymbol{R}_{S^2} & \text{Sp symmetric} \\ \left(\boldsymbol{R}_{RP^2} + \frac{1}{2}\frac{d}{dz}\right)\boldsymbol{R}_{S^2} & \text{Sp antisymmetric} \end{cases} \\
\left[W'\boldsymbol{R}_{D^2} + \mathrm{tr}(m'\boldsymbol{M}_{D^2})\right]_- &= \boldsymbol{R}_{D^2}\boldsymbol{R}_{S^2} \\
2\left[(\boldsymbol{M}_{D^2}m)_{ff'}\right]_- &= \boldsymbol{R}_{S^2}\delta_{ff'} \\
2\left[(m\boldsymbol{M}_{D^2})_{\tilde{f}\tilde{f}'}\right]_- &= \boldsymbol{R}_{S^2}\delta_{\tilde{f}\tilde{f}'}\,,
\end{aligned} \tag{3.10}$$

We separated the $\boldsymbol{R}_{RP^2}$ and $\boldsymbol{R}_{D^2}$ contributions using the difference in their dependence on $m_{\tilde{f}f}$ (see the argument below eq. (3.5)). Again, the last two equations are really the same

[3] In the SO antisymmetric and Sp symmetric cases, $\boldsymbol{R}_{RP^2}$ can be expressed [9, 8] in terms of $\boldsymbol{R}_{S^2}$, which leads to the expression

$$\mathcal{F}_{S^2}(\boldsymbol{S}) = \mp\frac{1}{2}\frac{\partial}{\partial\boldsymbol{S}}\mathcal{F}_{RP^2} \tag{3.9}$$

in the SO and Sp cases, respectively.

equation. For comparison, the U(N) loop equations are

$$\begin{aligned}
[W'\boldsymbol{R}_{S^2}]_- &= (\boldsymbol{R}_{S^2})^2 \\
[W'\boldsymbol{R}_{D^2} + \mathrm{tr}(m'\boldsymbol{M}_{D^2})]_- &= 2\boldsymbol{R}_{D^2}\boldsymbol{R}_{S^2} \\
[(\boldsymbol{M}_{D^2}m)_{ff'}]_- &= \boldsymbol{R}_{S^2}\delta_{ff'} \\
[(m\boldsymbol{M}_{D^2})_{\tilde{f}\tilde{f}'}]_- &= \boldsymbol{R}_{S^2}\delta_{\tilde{f}\tilde{f}'} \,.
\end{aligned} \tag{3.11}$$

Note that there is no RP^2 contribution for U(N).

The solutions to equations (3.10) or (3.11) are determined uniquely given the condition

$$\boldsymbol{S} = \oint_C \frac{dz}{2\pi i}\boldsymbol{R}_{S^2}(z)\,, \qquad 0 = \oint_C \frac{dz}{2\pi i}\boldsymbol{R}_{RP^2}(z)\,, \qquad 0 = \oint_C \frac{dz}{2\pi i}\boldsymbol{R}_{D^2}(z)\,. \tag{3.12}$$

In this sense, the loop equations are all we need to determine the free energy $\boldsymbol{F}$.

4. Connection between gauge theory and matrix model resolvents

On the gauge theory side we have arrived at the loop equations (2.17). If we can solve these equations for the resolvents, in particular for $T(z)$, we will have sufficient data to determine the glueball superpotential $W_{\mathrm{eff}}(S)$ up to a coupling independent part. In [5], it was shown for U(N) with adjoint matter that the solution can be obtained with the help of an auxiliary matrix model. On the other hand, in [4, 7, 10, 11] it was proved by perturbative diagram expansion that, for U(N) and SO / Sp with two-index tensor matter, if one only inserts up to two field strength superfield $\mathcal{W}_\alpha$'s per index loop then the calculation of $W_{\mathrm{eff}}(S)$ reduces to matrix integrals.

However, there are a number of reasons to study further the relation between the gauge theory and matrix model loop equations. First, as pointed out in [10] (see also p.11 of [5], and [22]), there are subtleties in using chiral ring relations at order S^h and higher, where h is the dual Coxeter number of the gauge group, and these could be related to the discrepancies observed in [10]. Since traces of schematic form $\mathrm{Tr}[(\mathcal{W}_\alpha^2)^n]$ $(n \geq h)$ can be rewritten in terms of lower power traces at these orders, imposing chiral ring relations *before* using the equation of motion of S is not necessarily justified. So, it is important to clarify how this subtlety is treated in the Konishi anomaly approach. Second, as a practical matter, the anomaly approach is more efficient than the diagrammatic approach in the cases we studied.

So, let us adopt the following point of view (some related ideas were explored in [14]). Let us not assume the reduction to a matrix model *a priori*. Then the gauge theory resolvents R, T, and M are just unknown functions that enable us to determine the coupling dependent part of the glueball effective action. We do know that we can evaluate the perturbative contribution to them by Feynman diagrams, but we do not know whether they are affected by nonperturbative effects or whether they can be calculated using a matrix model. These resolvents satisfy the loop equations (2.17), and given the conditions (2.19), they are determined uniquely. Similarly, the matrix model resolvents $\boldsymbol{R}_{S^2}$, $\boldsymbol{R}_{RP^2}$, $\boldsymbol{R}_{D^2}$, and $\boldsymbol{M}_{D^2}$ are now just functions satisfying matrix model loop equations (3.10). If we impose the condition (3.12), these resolvents are also determined uniquely, and by definition can be evaluated in matrix model perturbation theory.

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Now, let us ask what the relation between the two sets of resolvents is. Actually it is simple: if we know the matrix model resolvents, we can construct the gauge theory resolvents as follows. In the SO / Sp case,

$$\begin{aligned} R(z) &= \boldsymbol{R}_{S^2}(z)\,, \\ T(z) &= N\frac{\partial}{\partial \boldsymbol{S}}\boldsymbol{R}_{S^2}(z) + 4\boldsymbol{R}_{RP^2}(z) + \boldsymbol{R}_{D^2}(z)\,, \\ M(z) &= \boldsymbol{M}_{D^2}(z) \end{aligned} \tag{4.1}$$

with S and $\boldsymbol{S}$ identified; in the U(N) case, we get

$$\begin{aligned} R(z) &= \boldsymbol{R}_{S^2}(z)\,, \qquad w_\alpha(z) = w_\alpha\frac{\partial}{\partial \boldsymbol{S}}\boldsymbol{R}_{S^2}(z)\,, \\ T(z) &= N\frac{\partial}{\partial \boldsymbol{S}}\boldsymbol{R}_{S^2}(z) + \frac{w_\alpha w^\alpha}{2}\frac{\partial^2}{\partial \boldsymbol{S}^2}\boldsymbol{R}_{S^2}(z) + \boldsymbol{R}_{D^2}(z)\,, \\ M(z) &= \boldsymbol{M}_{D^2}(z) \end{aligned} \tag{4.2}$$

with the same $S = \boldsymbol{S}$ identification.[4] One can easily check that if the matrix model resolvents satisfy the matrix model loop equations (3.10) or (3.11), then the gauge theory resolvents satisfy the gauge theory loop equations (2.17) or (2.18). The requirement (2.19) is also satisfied provided that the matrix model resolvents satisfy the requirement (3.12). Further, these relations lead to

$$\langle \mathrm{Tr}[\Phi^p]\rangle_{\text{gauge theory}} = p\frac{\partial}{\partial g_p}W_{\text{eff}} = p\frac{\partial}{\partial g_p}\left[N\frac{\partial}{\partial S}\mathcal{F}_{S^2} + \frac{w_\alpha w^\alpha}{2}\frac{\partial^2}{\partial S^2}\mathcal{F}_{S^2} + 4\mathcal{F}_{RP^2} + \mathcal{F}_{D^2}\right], \tag{4.3}$$

which implies a relation between the effective superpotential and the matrix model quantities:

$$W_{\text{eff}} = N\frac{\partial}{\partial S}\mathcal{F}_{S^2} + \frac{w_\alpha w^\alpha}{2}\frac{\partial^2}{\partial S^2}\mathcal{F}_{S^2} + 4\mathcal{F}_{RP^2} + \mathcal{F}_{D^2} \tag{4.4}$$

up to a coupling independent additive part. This proves that the gauge theory diagrams considered in the Konishi anomaly approach reduce to matrix model integrals for all matter representations considered. Further, we do not have to take into account nonperturbative effects, since we can assume a perturbative expansion in the matrix model (although, strictly speaking, one should also verify that the Konishi anomalies receive no nonperturbative corrections [14]).

The relations (4.1) and (4.2) are consistent with inserting at most two $\mathcal{W}_\alpha$'s per index loop, but not with inserting more than two and then using Lie algebra relations. For instance, this can be seen from the diagrammatic expansion of $\boldsymbol{R}_{S^2}(z)$. So this shows us explicitly which diagrams are being computed in the Konishi anomaly approach.

In the U(N) case [5], it was convenient to collect all the gauge theory resolvents into a "superfield" $\mathcal{R}$, because of the "supersymmetry" under a shift of $\mathcal{W}_\alpha$ by a Grassmann number, and one could relate $\mathcal{R}$ to the matrix model resolvent $\boldsymbol{R}_{S^2}$. This fact enabled one to extract all the gauge theory resolvents solely from $\boldsymbol{R}_{S^2}$. However, in more general cases this trick does not work, and we have to relate the two sets of resolvents directly as in (4.1).

[4] Some of these relations have been written down in [18, 19, 20].

5. Traceless cases

So far, we considered two-index traceful matter Φ_{ij}, and discussed the relation between the gauge theory and the corresponding matrix model. In this section, we consider traceless[5] tensors $\tilde{\Phi}_{ij}$. These traceless tensors were studied in [10], and a method of evaluating the glueball effective superpotential $\tilde{W}_{\text{eff}}(S)$ from the combinatorics of the matrix model diagrams was given. However, the precise connection between the gauge theory and the matrix model quantities was not transparent, since one had to keep some of the matrix model diagrams and drop others in a way that seemed rather arbitrary from the matrix model point of view. Instead, here we show that the calculation of $\tilde{W}_{\text{eff}}(S)$ in gauge theory with *traceless* matter reduces to a *traceful* matrix model.[6]

5.1 Traceless gauge theory vs. traceful matrix model

To derive the generalized Konishi anomaly equation for a traceless tensor we have to use the appropriate projector

$$\tilde{P}_{ij,kl} \equiv P_{ij,kl} - \frac{1}{N}\delta_{ij}P_{mm,kl} = P_{ij,kl} - \frac{1}{N}\delta_{ij}\delta_{kl}, \tag{5.1}$$

where P is the projector of the corresponding traceful theory; the second equality holds for any projector defined in (2.14). The anomaly term (the second term of eq. (2.13)) is the same as in the traceful case, since the trace part is a singlet and does not couple to the gauge field. Therefore, the only difference in the anomaly equation between traceful and traceless cases is in the classical variation (the first term of eq. (2.13)), namely

$$\operatorname{Tr}\left[(Pf)\tilde{W}'(\Phi)\right] \to \operatorname{Tr}\left[(\tilde{P}f)\tilde{W}'(\Phi)\right] = \operatorname{Tr}\left[(Pf)\tilde{W}'(\Phi)\right] - \frac{1}{N}\operatorname{Tr}[f]\operatorname{Tr}\left[\tilde{W}'(\Phi)\right]. \tag{5.2}$$

For definiteness, let us focus on SU(N) adjoint matter, which can be thought of as traceless U(N) adjoint matter, without fundamentals added; we will generalize the discussion to other groups and matter representations afterward. In this case, the last term of eq. (5.2) changes the U(N) loop equation (the first and the third lines of (2.18)) to

$$\left[\tilde{W}'(z)\tilde{R}(z)\right]_- + g_1\tilde{R}(z) = \tilde{R}(z)^2, \qquad \left[\tilde{W}'(z)\tilde{T}(z)\right]_- + g_1\tilde{T}(z) = 2\tilde{R}(z)\tilde{T}(z)\,. \tag{5.3}$$

Note that $w_\alpha(z) = 0$ for SU(N). The constant g_1 is

$$g_1 \equiv -\frac{1}{N}\left\langle \operatorname{Tr}\left[\tilde{W}'(\tilde{\Phi})\right]\right\rangle. \tag{5.4}$$

If we define

$$W(z) \equiv \tilde{W}(z) + g_1 z\,, \tag{5.5}$$

[5] In this section, we denote traceless quantities by tildes to distinguish them from their traceful counterparts.

[6] A connection between traceful and traceless gauge theories in the Leigh-Strassler deformed $\mathcal{N} = 4$ SU(N) theory was discussed in [25].

the above equations are

$$[W'(z)\tilde{R}(z)]_- = \tilde{R}(z)^2\,, \qquad [W'(z)\tilde{T}(z)]_- = 2\tilde{R}(z)\tilde{T}(z)\,. \tag{5.6}$$

These are of the same form as the loop equations with *traceful* matter and the tree level superpotential W. Therefore, in order to obtain the effective glueball superpotential $\tilde{W}_{\rm eff}(S)$ for traceless matter, we can instead solve the traceful theory with the shifted tree level superpotential W, choosing the value of g_1 appropriately. The solution to these loop equations is determined uniquely given the condition

$$S = \oint_C \frac{dz}{2\pi i}\tilde{R}(z)\,, \qquad N = \oint_C \frac{dz}{2\pi i}\tilde{T}(z)\,. \tag{5.7}$$

In the case of traceful matter, the contour is around a critical point of the tree level superpotential. However, for traceless matter, the loop equations above tell us that the contour should be taken around the critical point of the shifted superpotential (5.5), rather than the original $\tilde{W}$. This is because we cannot change all the eigenvalues of $\tilde{\Phi}$ independently due to the tracelessness condition $\mathrm{Tr}[\tilde{\Phi}]=0$.

Let the resolvents of the traceful theory with tree level superpotential $W(\Phi)$ be R and T, with g_1 treated as an independent variable. R and T are functions of z, $g_{p\geq 1}$ as well as S, N: $R = R(z;g_{p\geq 1},S)$, $T = T(z;g_{p\geq 1},S,N)$. We will often omit S and N in the arguments henceforth to avoid clutter. Since R and T satisfy the same loop equations as $\tilde{R}$ and $\tilde{T}$ provided g_1 is chosen appropriately, i.e. $g_1 = g_1(g_{p\geq 2},S,N)\equiv \tilde{g}_1$, it should be that

$$\tilde{R}(z;g_{p\geq 2}) = R(z;g_{p\geq 1})\big|_{g_1=\tilde{g}_1}\,, \qquad \tilde{T}(z;g_{p\geq 2}) = T(z;g_{p\geq 1})\big|_{g_1=\tilde{g}_1}\,. \tag{5.8}$$

These satisfy the conditions (5.7) given that R and T satisfy the conditions (5.7) without tildes. Expanding these in z, we find

$$\left\langle \mathrm{Tr}[\mathcal{W}^2\tilde{\Phi}^p]\right\rangle^{\rm traceless}_{g_{p\geq 2}} = \left\langle \mathrm{Tr}[\mathcal{W}^2\Phi^p]\right\rangle^{\rm traceful}_{g_{p\geq 1}}\Big|_{g_1=\tilde{g}_1}\,, \quad \left\langle \mathrm{Tr}[\tilde{\Phi}^p]\right\rangle^{\rm traceless}_{g_{p\geq 2}} = \left\langle \mathrm{Tr}[\Phi^p]\right\rangle^{\rm traceful}_{g_{p\geq 1}}\Big|_{g_1=\tilde{g}_1}\,. \tag{5.9}$$

In particular, setting $p=1$ in the second equation,

$$\langle \mathrm{Tr}[\Phi]\rangle^{\rm traceful}_{g_{p\geq 1}}\big|_{g_1=\tilde{g}_1} = \left[\frac{\partial}{\partial g_1}T(z;g_{p\geq 1})\right]\Bigg|_{g_1=\tilde{g}_1} = 0\,, \tag{5.10}$$

which can be used for determining g_1 in terms of all other parameters.[7] We infer from eq. (5.8) equality between the traceless and traceful effective superpotentials:

$$\tilde{W}_{\rm eff}(g_{p\geq 2},S,N) = W_{\rm eff}(g_{p\geq 1},S,N)|_{g_1=\tilde{g}_1(g_{p\geq 2},S,N)}\,. \tag{5.11}$$

As long as we impose the tracelessness condition (5.10), this correctly reproduces the relation (5.9). Note that $\tilde{g}_1$ depends on N; this is the origin of the complicated N dependence of $\tilde{W}_{\rm eff}$ found in [10].

[7]One might have expected that g_1 can be determined by eq. (5.4). However, it is easy to show using the relation (5.9) that the equation is just the equation of motion of the traceful theory, which is identically satisfied for any g_1: $0\equiv \langle \mathrm{Tr}[W'(\Phi)]\rangle = \langle \mathrm{Tr}[\tilde{W}'(\tilde{\Phi})]\rangle + Ng_1$.

Because we know that the traceful theory can be solved by the associated traceful matrix model, we can calculate the effective superpotential using that matrix model. Specifically, in the present case, it is given in terms of the free energy of the traceful matrix model by

$$\tilde{W}_{\rm eff}(g_{p\geq 2}, S, N) = \left[N \frac{\partial}{\partial S} \mathcal{F}_{S^2} \right] \Bigg|_{g_1 = \tilde{g}_1} . \tag{5.12}$$

The function $\tilde{g}_1(g_2, g_3, \ldots, S, N)$ is determined by

$$\langle {\rm Tr}[\tilde{\Phi}] \rangle = \left[N \frac{\partial}{\partial S} \frac{\partial}{\partial g_1} \mathcal{F}_{S^2} \right] \Bigg|_{g_1 = \tilde{g}_1} = 0 \,. \tag{5.13}$$

If we add fundamental fields, the shift constant g_1 is changed to

$$g_1 \equiv -\frac{1}{N} \left\langle {\rm Tr}[\tilde{W}'(\tilde{\Phi})] \right\rangle - \frac{1}{N} \left\langle \tilde{Q}_{\tilde{f}} m_{\tilde{f} f} Q_f \right\rangle , \tag{5.14}$$

but everything else remains the same; we just have to work with the traceful theory and the shifted tree level superpotential. g_1 is determined by the tracelessness condition.

We only discussed the $\mathrm{SU}(N)$ case in the above, but the generalization to other tensors, i.e., SO traceless symmetric tensor and Sp traceless antisymmetric tensor, is straightforward. We just shift the tree level superpotential as (5.5), and work with the traceful theory instead.

5.2 Examples

Here we explicitly demonstrate how the method outlined above works in the case of a cubic tree level superpotential,

$$\tilde{W}(\tilde{\Phi}) = \frac{m}{2} \tilde{\Phi}^2 + \frac{g}{3} \tilde{\Phi}^3 \,. \tag{5.15}$$

The associated traceful tree level superpotential is

$$W(\Phi) = \lambda \Phi + \frac{m}{2} \Phi^2 + \frac{g}{3} \Phi^3 \tag{5.16}$$

$(g_1 = \lambda, g_2 = m, g_3 = g)$.

5.2.1 $\mathrm{SU}(N)$ adjoint

We first consider $\mathrm{SU}(N)$ with adjoint matter and no fundamentals. In [10] it was found by perturbative computation to order g^6 that the corresponding $W_{\rm eff}$ vanishes due to a cancellation among diagrams. We will now prove that $W_{\rm eff} = 0$ to all orders in g.

The planar contributions to the free energy of the traceful matrix model can be computed exactly by the standard method [23]:

$$\mathcal{F}_{S^2} = S W_0 + \frac{1}{2} S^2 \ln \left(\frac{\tilde{m}}{\sqrt{1+y}\, m} \right) - \frac{2}{3} \frac{S^2}{y} \left[1 + \frac{3}{2} y + \frac{1}{8} y^2 - (1+y)^{3/2} \right] \tag{5.17}$$

with

$$\tilde{m} = \sqrt{m^2 - 4\lambda g}$$

$$
\begin{aligned}
W_0 &= \frac{1}{2g}(\tilde{m}-m)\left(\lambda + \frac{1}{12g}(\tilde{m}-m)(\tilde{m}+2m)\right) \\
\frac{y}{(1+y)^{3/2}} &= \frac{8g^2 S}{m^3}\,. \qquad (5.18)
\end{aligned}
$$

We discarded some g independent contributions. The W_0 term arises from shifting Φ to eliminate the linear term in $W(\Phi)$. The superpotential is therefore

$$
W_{\text{eff}} = N\frac{\partial \mathcal{F}_{S^2}}{\partial S} = NW_0 + \frac{NS}{6y}\left[-4 - 6y + 6y\ln\left(\frac{\tilde{m}}{m\sqrt{1+y}}\right) + 4(1+y)^{3/2}\right]. \qquad (5.19)
$$

Imposing $\partial W_{\text{eff}}/\partial\lambda = 0$ leads to, after some algebra,

$$
\lambda = -\frac{2gS}{m}\,, \qquad y = \frac{8g^2 S}{m^3}\,. \qquad (5.20)
$$

Substituting back into (5.19) and doing some more algebra, we find

$$
W_{\text{eff}} = 0\,. \qquad (5.21)
$$

This vanishing of the perturbative contribution to the effective superpotential is consistent with the gauge theory analysis of [15]. In fact, it is shown there that $W_{\text{eff}} = 0$ for any tree level superpotential with only odd power interactions.

5.2.2 Sp(N) antisymmetric tensor

Now consider Sp(N) with an antisymmetric tensor and no fundamentals. By diagram calculations or by computer, the planar and RP^2 contributions to the free energy of the traceful matrix model are

$$
\begin{aligned}
\mathcal{F}_{S^2} &= -\frac{\lambda^2 S}{2m} - \left(\frac{\lambda S^2}{2m^2} + \frac{\lambda^3 S}{3m^3}\right)g - \left(\frac{S^3}{6m^3} + \frac{\lambda^2 S^2}{m^4} + \frac{\lambda^4 S}{2m^3}\right)g^2 - \\
&\quad - \left(\frac{\lambda S^3}{m^5} + \frac{8\lambda^3 S^2}{3m^6} + \frac{\lambda^5 S}{m^7}\right)g^3 - \left(\frac{S^4}{3m^6} + \frac{5\lambda^2 S^3}{m^7} + \frac{8\lambda^4 S^2}{m^8} + \frac{7\lambda^6 S}{3m^9}\right)g^4 - \\
&\quad - \left(\frac{4\lambda S^4}{m^8} + \frac{70\lambda^3 S^3}{3m^9} + \frac{128\lambda^5 S^2}{5m^{10}} + \frac{6\lambda^7 S}{m^{11}}\right)g^5 - \\
&\quad - \left(\frac{7S^5}{6m^9} + \frac{32\lambda^2 S^4}{m^{10}} + \frac{105\lambda^4 S^3}{m^{11}} + \frac{256\lambda^6 S^2}{3m^{12}} + \frac{33\lambda^8 S}{2m^{13}}\right)g^6 - \\
&\quad - \left(\frac{21\lambda S^5}{m^{11}} + \frac{640\lambda^3 S^4}{3m^{12}} + \frac{462\lambda^5 S^3}{m^{13}} + \frac{2048\lambda^7 S^2}{7m^{14}} + \frac{143\lambda^9 S}{3m^{15}}\right)g^7 - \\
&\quad - \left(\frac{16S^6}{3m^{12}} + \frac{231\lambda^2 S^5}{m^{13}} + \frac{1280\lambda^4 S^4}{m^{14}} + \frac{2002\lambda^6 S^3}{m^{15}} + \frac{1024\lambda^8 S^2}{m^{16}} + \frac{143\lambda^{10} S^1}{m^{17}}\right)\times \\
&\quad \times g^8 + \mathcal{O}(g^9)\,, \qquad (5.22) \\
\mathcal{F}_{RP^2} &= \frac{\lambda S}{2m^2}g + \left(\frac{3\lambda S^2}{8m^3} + \frac{\lambda^2 S}{m^4}\right)g^2 + \left(\frac{9\lambda S^2}{4m^5} + \frac{8\lambda^3 S}{3m^6}\right)g^3 +
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{59S^3}{48m^6}+\frac{45\lambda^2S^2}{4m^7}+\frac{8\lambda^4S}{m^8}\right)g^4+\left(\frac{59\lambda S^3}{4m^8}+\frac{105\lambda^3S^2}{2m^9}+\frac{128\lambda^5S}{5m^{10}}\right)g^5+\\
&+\left(\frac{197S^4}{32m^9}+\frac{118\lambda^2S^3}{m^{10}}+\frac{945\lambda^4S^2}{4m^{11}}+\frac{256\lambda^6S}{3m^{12}}\right)g^6+\\
&+\left(\frac{1773\lambda S^4}{16m^{11}}+\frac{2360\lambda^3S^3}{3m^{12}}+\frac{2079\lambda^5S^2}{2m^{13}}+\frac{2048\lambda^7S}{7m^{14}}\right)g^7+\\
&+\left(\frac{4775S^5}{128m^{12}}+\frac{19503\lambda^2S^4}{16m^{13}}+\frac{4720\lambda^4S^3}{m^{14}}+\frac{9009\lambda^6S^2}{2m^{15}}+\frac{1024\lambda^8S}{m^{16}}\right)\times\\
&\times g^8+\mathcal{O}(g^9)\,,
\end{aligned}
\tag{5.23}
$$

up to a λ and g independent part. From the tracelessness (5.10), we find

$$
\begin{aligned}
\lambda=&\left(-1+\frac{2}{N}\right)\frac{S}{m}g+\left(-\frac{3}{N}+\frac{12}{N^2}\right)\frac{S^2}{m^4}g^3+\left(-\frac{1}{N}-\frac{24}{N^2}+\frac{160}{N^3}\right)\frac{S^3}{m^7}g^5+\\
&+\left(-\frac{3}{4N}-\frac{27}{N^2}-\frac{192}{N^3}+\frac{2688}{N^4}\right)\frac{S^4}{m^{10}}g^7+\mathcal{O}(g^9)\equiv\tilde\lambda\,.
\end{aligned}
\tag{5.24}
$$

Therefore, the effective superpotential is, up to an α independent additive part,

$$
\begin{aligned}
\tilde W_{\rm eff}&=W_{\rm eff}|_{\lambda=\tilde\lambda}=\left[N\frac{\partial}{\partial S}\mathcal{F}_{S^2}+4\mathcal{F}_{RP^2}\right]\Big|_{\lambda=\tilde\lambda}\\
&=\left(-1+\frac{4}{N}\right)S^2\alpha+\left(-\frac{1}{3}-\frac{8}{N}+\frac{160}{3N^2}\right)S^3\alpha^2+\left(-\frac{1}{3}-\frac{12}{N}-\frac{256}{3N^2}+\frac{3584}{3N^3}\right)S^4\alpha^3+\\
&\quad+\left(-\frac{1}{2}-\frac{24}{N}-\frac{352}{N^2}+\frac{33792}{N^4}\right)S^5\alpha^4+\cdots,
\end{aligned}
\tag{5.25}
$$

where $\alpha\equiv\frac{g^2}{2m^3}$. This reproduces the result of [10] up to $\mathcal{O}(\alpha^3)$ and extends it further to $\mathcal{O}(\alpha^4)$.

From these examples, the advantage of the present approach over the traceless diagram approach of [10] should be clear. In that approach, one has to evaluate contributing diagrams order by order and evaluating the combinatorics gets very cumbersome. On the other hand, in this traceful approach, there is no issue of keeping and dropping diagrams, and calculations can be done more systematically. Therefore, being able to reduce the traceless problem to a traceful problem is a great advantage.

5.3 Traceless matrix model

We saw that the traceless gauge theory can be solved by the traceful matrix model, not the traceless matrix model. In the following, we argue that the traceless matrix model is not useful in determining the effective superpotential of the traceless gauge theory, $\tilde W_{\rm eff}$. The relation among traceless and traceful theories, as far as the effective superpotential is concerned, is shown in figure 1.

The matrix model loop equation for traceless matter can be derived almost in parallel to the traceless gauge theory loop equation derived in the previous subsection. Again, we replace the projector P with the appropriate traceless version $\tilde P$. For example, in the case of SU(N) adjoint without fundamentals, which was considered in the previous section on the gauge theory side, the loop equation is

$$\left[W'\tilde{\boldsymbol{R}}_{S^2}\right]_- = \left(\tilde{\boldsymbol{R}}_{S^2}\right)^2 . \tag{5.26}$$

Here W is the shifted superpotential defined in (5.5), with g_1 defined in (5.4) and the gauge theory expectation values replaced by the matrix model expectation values.

Eq. (5.26) is of the same form as the traceful matrix model loop equation, and the first equation of the traceless gauge theory loop equations (5.6). Finally, using the equivalence of the traceful gauge theory and matrix model, we conclude that

$$\tilde{\boldsymbol{R}}(z;g_{p\geq 2}) = \boldsymbol{R}(z;g_{p\geq 1})\big|_{g_1=\tilde{g}_1} = R(z;g_{p\geq 1})\big|_{g_1=\tilde{g}_1} = \tilde{R}(z;g_{p\geq 2}) . \tag{5.27}$$

However, what we need to determine $\tilde{W}_{\text{eff}}$ is $\tilde{T}$, which we saw in the last subsection to be obtainable from the traceful theory as

$$\tilde{T}(z;g_{p\geq 2},S,N) = T(z;g_{p\geq 2},S,N)\big|_{g_1=\tilde{g}_1(g_2,g_3,\cdots,S,N)} = \left[N\frac{\partial}{\partial S}R(z;g_{p\geq 2})\right]\Bigg|_{g_1=\tilde{g}_1(g_2,g_3,\cdots,S,N)} . \tag{5.28}$$

From the standpoint of the traceless matrix model, the only thing we know is $\tilde{\boldsymbol{R}} = \tilde{R} = R|_{g_1=\tilde{g}_1}$, and we have no information about the g_1 dependence of R. In the framework of the traceless matrix model, there is no way of performing the derivative $\partial/\partial S$ in (5.28) before making the replacement $g_1 = \tilde{g}_1$, because $\tilde{g}_1$ depends on S also.

Therefore, it is impossible to obtain the effective superpotential for the traceless gauge theory directly, just by using the data from the corresponding traceless matrix model. We really need to invoke the traceful matrix model.

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A. Loop equations on the gauge theory side

In this appendix, we are going to calculate the gauge theory loop equations using the approach of [5, 16]. We start with generalized Konishi currents and corresponding transformations of the fields

$$\begin{aligned} J_f &\equiv \operatorname{Tr}\Phi^\dagger e^{V_{\text{adj}}} f(\mathcal{W}_\alpha,\Phi) \quad &\Rightarrow \quad \delta\Phi &= f(\mathcal{W}_\alpha,\Phi) \\ J_g &\equiv Q_f^\dagger e^{V_{\text{fund}}} g_{ff'}(\Phi)Q_{f'} \quad &\Rightarrow \quad \delta Q_f &= g_{ff'}(\Phi)Q_{f'} . \end{aligned} \tag{A.1}$$

$$\begin{array}{ccc} \boxed{\text{traceful gauge theory}} & \leftrightarrow & \boxed{\text{traceful matrix model}} \\ \downarrow & & \downarrow \\ \boxed{\text{traceless gauge theory}} & \not\leftrightarrow & \boxed{\text{traceless matrix model}} \end{array}$$

Figure 1: Relation among traceful and traceless theories.

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The explicitly written indices on the Q_f's and $g_{ff'}$ are flavor indices, and gauge indices are suppressed. We find the generalized anomaly equations

$$\begin{aligned}\bar{D}^2 J_f &= \operatorname{Tr} f(\mathcal{W}_\alpha, \Phi) W'(\Phi) + \tilde{Q} f(\mathcal{W}_\alpha, \Phi) m'(\Phi) Q + \sum_{jklm} A_{jk,lm} \frac{\partial f_{kj}}{\partial \Phi_{lm}} \\ \bar{D}^2 J_g &= 2\tilde{Q} m(\Phi) g(\Phi) Q + \operatorname{Tr} A^{\text{fund}} g(\Phi)\end{aligned} \tag{A.2}$$

and $\bar{D}^2 J_f$ and $\bar{D}^2 J_g$ vanish in the chiral ring.

The field Φ being considered transforms by commutation under gauge transformations, so the elementary anomaly coefficient is the same as the one appearing in [5],

$$\begin{aligned}A_{jk,lm} &= \frac{1}{32\pi^2} [(\mathcal{W}_\alpha \mathcal{W}^\alpha)_{jm} \delta_{lk} + (\mathcal{W}_\alpha \mathcal{W}^\alpha)_{lk} \delta_{jm} - 2(\mathcal{W}_\alpha)_{jm} (\mathcal{W}^\alpha)_{lk}] \\ &\equiv \frac{1}{32\pi^2} \{\mathcal{W}_\alpha, [\mathcal{W}^\alpha, e_{ml}]\}_{jk}\end{aligned} \tag{A.3}$$

where e_{ml} is the basis matrix with the single non-zero entry $(e_{ml})_{jk} = \delta_{mj}\delta_{lk}$. For fields transforming in the fundamental representation we should use

$$A^{\text{fund}}_{jk} = \frac{1}{32\pi^2} (\mathcal{W}_\alpha \mathcal{W}^\alpha)_{jk} \tag{A.4}$$

There is one modification in the treatment of fundamental fields, as compared to the U(N) case studied in [16]. Since the fundamental representation is real for SO and pseudo-real for Sp, the fields Q and $\tilde{Q}$ are not independent; instead, they are related by (2.4). This results in the factor of 2 in the second equation in (A.2), but otherwise the discussion proceeds as in [16]. In the rest of the appendix we omit reference to fundamentals.

Next we consider the symmetries of Φ. In equation (A.1), $f = \delta\Phi$ must have the same symmetry properties as Φ itself. The tensor field will be taken either symmetric or antisymmetric. We can discuss all four cases in a uniform fashion by using the notation

$$\Phi^T = \begin{cases} \sigma\Phi & \text{for groups } \mathrm{SO}(N)\,, \\ \sigma J \Phi J^{-1} & \text{for groups } \mathrm{Sp}(N)\,, \end{cases} \tag{A.5}$$

and $\sigma = \pm 1$. The gauge field satisfies ${\mathcal{W}_\alpha}^T = -\mathcal{W}_\alpha$ for SO groups, and ${\mathcal{W}_\alpha}^T = -J\mathcal{W}_\alpha J^{-1}$ for Sp groups. As discussed in Subsection 2.2, Φ has the property

$$\Phi = P\Phi\,, \quad \text{or explicitly} \quad \Phi_{ab} = P_{ab,ij}\Phi_{ij} \tag{A.6}$$

with the projectors defined in (2.14). To ensure that f has the same symmetry as Φ, we should replace $f \to Pf$ in (A.2). Specifically, we will take $\delta\Phi$ of the form

$$f^{SO} = P^{SO} \frac{B}{z - \Phi} = \left(\frac{B}{z-\Phi}\right) + \sigma \left(\frac{B}{z-\Phi}\right)^T = \left(\frac{B}{z-\Phi}\right) + \sigma \left(\frac{B^T}{z - \sigma\Phi}\right), \tag{A.7}$$

$$f^{Sp} = P^{Sp} \frac{B}{z - \Phi} = \left(\frac{B}{z-\Phi}\right) + \sigma J \left(\frac{B}{z-\Phi}\right)^T J = \left(\frac{B}{z-\Phi}\right) + \sigma \left(\frac{JB^TJ}{z - \sigma\Phi}\right), \tag{A.8}$$

with $B = 1$ or $B = \mathcal{W}^2 \equiv \mathcal{W}_\beta \mathcal{W}^\beta$. Using the symmetry of the gauge field and the chiral ring relations, both (A.7) and (A.8) reduce to

$$f = \frac{B}{z - \Phi} + \sigma \frac{B}{z - \sigma\Phi} \,. \tag{A.9}$$

Also, to take derivatives with respect to matrix elements[8] correctly we should set

$$\partial_{lm} \Phi_{ab} = P_{lm,ab} \,. \tag{A.10}$$

Then the tensor field anomaly term becomes

$$\begin{aligned} A_{jk,lm} \partial_{lm} f_{kj} \; = \; & \frac{1}{32\pi^2} \left[(\mathcal{W}^2)_{jm} \delta_{lk} + \delta_{jm} (\mathcal{W}^2)_{lk} - 2 (\mathcal{W}_\alpha)_{jm} (\mathcal{W}^\alpha)_{lk} \right] \times \\ & \times \left[\left(\frac{B}{z - \Phi} \right)_{ra} \left(\frac{1}{z - \Phi} \right)_{bs} P_{kj,rs} P_{lm,ab} \right] . \end{aligned} \tag{A.11}$$

After using the projectors (2.14), the identity $\mathrm{Tr}\, \mathcal{W}_\alpha \Phi^k = 0$, the symmetry properties of Φ and $\mathcal{W}_\alpha$, and the chiral ring relations, we find

$$\begin{aligned} A_{jk,lm} \partial_{lm} f_{kj} \; = \; & \frac{1}{32\pi^2} \Bigg[\left(\mathrm{Tr}\, \frac{\mathcal{W}^2}{z - \Phi} \right) \left(\mathrm{Tr}\, \frac{B}{z - \Phi} \right) + \left(\mathrm{Tr}\, \frac{1}{z - \Phi} \right) \left(\mathrm{Tr}\, \frac{\mathcal{W}^2 B}{z - \Phi} \right) + \\ & + 4k\sigma \;\; \mathrm{Tr}\, \frac{\mathcal{W}^2 B}{(z - \Phi)(z - \sigma\Phi)} \Bigg] \end{aligned} \tag{A.12}$$

The only difference in (A.12) between the two types of gauge groups is that the sign in front of the single trace term is $k = +1$ for SO, and $k = -1$ for Sp. Taking $B = 1$ and $B = \mathcal{W}^2$ in (A.12) we find

$$\begin{aligned} 0 \; = \; & \mathrm{Tr}\, \frac{W'(\Phi)}{z - \Phi} + \sigma\, \mathrm{Tr}\, \frac{W'(\Phi)}{z - \sigma\Phi} + \\ & + \frac{2}{32\pi^2} \left[\left(\mathrm{Tr}\, \frac{\mathcal{W}^2}{z - \Phi} \right) \left(\mathrm{Tr}\, \frac{1}{z - \Phi} \right) + 2k\sigma \left(\mathrm{Tr}\, \frac{\mathcal{W}^2}{(z - \Phi)(z - \sigma\Phi)} \right) \right] \end{aligned} \tag{A.13}$$

$$0 \; = \; \mathrm{Tr}\, \frac{\mathcal{W}^2 W'(\Phi)}{z - \Phi} + \sigma\, \mathrm{Tr}\, \frac{\mathcal{W}^2 W'(\Phi)}{z + \Phi} + \frac{1}{32\pi^2} \left[\left(\mathrm{Tr}\, \frac{\mathcal{W}^2}{z - \Phi} \right) \left(\mathrm{Tr}\, \frac{\mathcal{W}^2}{z - \Phi} \right) \right] \tag{A.14}$$

Now recall that $W(\Phi)^T = W(\Phi)$ for $\mathrm{SO}(N)$, and $W(\Phi)^T = J W(\Phi) J^{-1}$ for $\mathrm{Sp}(N)$ since it only appears inside a trace; so

$$\mathrm{Tr}\, \frac{W'(\Phi)}{z - \sigma\Phi} = \sigma\, \mathrm{Tr}\, \frac{W'(\Phi)}{z - \Phi} \,, \qquad \mathrm{Tr}\, \frac{\mathcal{W}^2 W'(\Phi)}{z - \sigma\Phi} = \sigma\, \mathrm{Tr}\, \frac{\mathcal{W}^2 W'(\Phi)}{z - \Phi} \,. \tag{A.15}$$

The single trace terms have to be treated separately: when $\sigma = -1$,

$$\mathrm{Tr}\, \frac{\mathcal{W}^2}{z^2 - \Phi^2} = \frac{1}{2z} \;\; \mathrm{Tr} \left[\mathcal{W}^2 \left(\frac{1}{z - \Phi} + \frac{1}{z + \Phi} \right) \right] = \frac{1}{z} \;\; \mathrm{Tr}\, \frac{\mathcal{W}^2}{z - \Phi} \tag{A.16}$$

while for $\sigma = +1$, we should use

$$\mathrm{Tr}\, \frac{\mathcal{W}^2}{(z - \Phi)^2} = -\frac{d}{dz} \;\; \mathrm{Tr}\, \frac{\mathcal{W}^2}{z - \Phi} \,. \tag{A.17}$$

Putting everything together, we find the loop equations written in equation (2.17).

[8] In the case of $\mathrm{U}(N)$ of [5], one had $P_{lm,ab} = (e_{lm})_{ab}$ which satisfies $(e_{lm})_{ab} B_{ab} = B_{lm}$, for any matrix B.

B. Loop equations on the matrix model side

Here we derive the matrix model loop equations for SO / Sp following Seiberg [16], who discussed the U(N) case. Start with the matrix model partition function

$$Z = \int d\mathbf{\Phi} d\mathbf{Q} \,\exp\left\{ -\frac{1}{\boldsymbol{g}} \left[\mathrm{Tr}[W(\mathbf{\Phi})] + \tilde{\mathbf{Q}}_{\tilde{f}} m_{\tilde{f}f}(\mathbf{\Phi}) \mathbf{Q}_f \right] \right\}. \tag{B.1}$$

Because the fundamental matter is real for SO(N) and pseudo-real for Sp(N), there is no integration over $\tilde{\mathbf{Q}}$. It is not an independent variable, but related to $\mathbf{Q}$ by eq. (2.4). We will write the symmetry properties of the the tensor field $\mathbf{\Phi}$ as

$$\mathbf{\Phi}^T = \begin{cases} \sigma \mathbf{\Phi} & \mathrm{SO}(N)\,, \\ \sigma J \mathbf{\Phi} J^{-1} & \mathrm{Sp}(N)\,. \end{cases} \tag{B.2}$$

where $\sigma = \pm 1$. The matrix $m(\mathbf{\Phi})$ has symmetry properties as given in eq. (2.7).

Now we perform two independent transformations

$$\delta \mathbf{\Phi} = BP \frac{1}{z - \mathbf{\Phi}}\,, \qquad \delta \mathbf{Q}_f = \lambda_{ff'} \frac{1}{z - \mathbf{\Phi}} \mathbf{Q}_{f'} \tag{B.3}$$

where B (number) and λ (matrix) are independent and infinitesimal. To make sure that $\delta\mathbf{\Phi}$ has the same symmetry properties as $\mathbf{\Phi}$ itself, we have introduced the appropriate projector P in (B.3), see eq. (2.14). The measure in (B.1) changes as

$$\begin{aligned} d\mathbf{\Phi} &\to d\mathbf{\Phi}\, J_{\mathbf{\Phi}} = d\mathbf{\Phi}\,(1 + \Delta_{\mathbf{\Phi}})\,, \\ d\mathbf{Q} &\to d\mathbf{Q}\, J_{\mathbf{Q}} = d\mathbf{Q}\,(1 + \Delta_{\mathbf{Q}}) \end{aligned} \tag{B.4}$$

to first order in B and λ, where the corresponding changes in the jacobians are

$$\begin{aligned} \Delta_{\mathbf{\Phi}} &= BP_{ij,ab} \left(\frac{1}{z - \mathbf{\Phi}} \right)_{ia} \left(\frac{1}{z - \mathbf{\Phi}} \right)_{bj} \\ \Delta_{\mathbf{Q}} &= \left(\mathrm{Tr}\, \frac{1}{z - \mathbf{\Phi}} \right) (\mathrm{tr}\,\lambda)\,, \end{aligned} \tag{B.5}$$

where tr is a trace over the flavor indices. The classical pieces change by

$$\delta\ \mathrm{Tr}[W(\mathbf{\Phi})] = B\ \mathrm{Tr} \left[W'(\mathbf{\Phi}) P \frac{1}{z - \mathbf{\Phi}} \right] = B\ \mathrm{Tr}\, \frac{W'(\mathbf{\Phi})}{z - \mathbf{\Phi}}\,. \tag{B.6}$$

One can show the second equality using symmetry properties of W: since it only enters Z in the form of the trace, we should take $W(\mathbf{\Phi}^T) = W(\mathbf{\Phi})$ for SO and $W(\mathbf{\Phi}^T) = JW(\mathbf{\Phi})J^{-1}$ for Sp. Similarly,

$$\begin{aligned} \delta(\tilde{\mathbf{Q}} m \mathbf{Q}) &= \tilde{\mathbf{Q}} \left(\frac{\lambda^T m}{z - \sigma\mathbf{\Phi}} + \frac{m\lambda}{z - \mathbf{\Phi}} \right) \mathbf{Q} + B \tilde{\mathbf{Q}} m' \left(P \frac{1}{z - \mathbf{\Phi}} \right) \mathbf{Q} \\ &= 2\tilde{\mathbf{Q}} \frac{m\lambda}{z - \mathbf{\Phi}} \mathbf{Q} + B \tilde{\mathbf{Q}} \frac{m'}{z - \mathbf{\Phi}} \mathbf{Q}\,, \end{aligned} \tag{B.7}$$

where we used a similar symmetry property of the matrix m. Finally, with the explicit form of the projectors (2.14) we find that in all four cases the statement $\delta Z = 0$ gives two independent loop equations (one for B, and one for λ):

$$\frac{1}{2}\left\langle\left(\boldsymbol{g}\,\mathrm{Tr}\,\frac{1}{z-\boldsymbol{\Phi}}\right)^2\right\rangle \pm \frac{\sigma}{2}\boldsymbol{g}\left\langle\boldsymbol{g}\,\mathrm{Tr}\,\frac{1}{(z-\boldsymbol{\Phi})(z-\sigma\boldsymbol{\Phi})}\right\rangle = \left\langle\boldsymbol{g}\,\mathrm{Tr}\,\frac{W'(\boldsymbol{\Phi})}{z-\boldsymbol{\Phi}}\right\rangle + \boldsymbol{g}\left\langle\tilde{\boldsymbol{Q}}\frac{m'(\boldsymbol{\Phi})}{z-\boldsymbol{\Phi}}\boldsymbol{Q}\right\rangle,$$
$$\left\langle\boldsymbol{g}\,\mathrm{Tr}\,\frac{1}{z-\boldsymbol{\Phi}}\right\rangle\delta_{ff'} = 2\left\langle\tilde{\boldsymbol{Q}}_{\tilde{f}}\frac{m_{\tilde{f}f}(\boldsymbol{\Phi})}{z-\boldsymbol{\Phi}}\boldsymbol{Q}_{f'}\right\rangle, \qquad \text{(B.8)}$$

for SO and Sp, respectively. This is eq. (3.8) quoted in section 3.

As it is written, equation (B.8) includes all orders in $\boldsymbol{g}$. The anomaly term in the first equation (B.8) factorizes as

$$\left\langle\left(\boldsymbol{g}\,\mathrm{Tr}\,\frac{1}{z-\boldsymbol{\Phi}}\right)^2\right\rangle = \left\langle\boldsymbol{g}\,\mathrm{Tr}\,\frac{1}{z-\boldsymbol{\Phi}}\right\rangle^2 \times \left[1+\mathcal{O}(\boldsymbol{g}^2)\right] \qquad \text{(B.9)}$$

as can be seen from a diagram expansion. With this and the definition of matrix model resolvents (3.3) and (3.5), we obtain the loop equations (3.10).

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